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Approximations Via Whittaker's Cardinal Function*

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Whittaker's cardinal function is used to derive various types of extremely accurate approximation procedures, along with error bounds, for interpolating, integrating, and evaluating the Fourier (over $(-\infty, \infty)$ only) and the Hilbert (over $(-\infty, \infty)$, $(0, \infty)$, and $(-1, 1)$ transforms of functions. Formulas over $(-\infty, \infty)$ are obtained directly; in practice these are especially suitable for functions that are analytic in the strip $\mathscr{D}_d' = \{x + iy: |y| \leqslant d\}$, $d > 0$, and which go to zero rapidly as $x \to \pm\infty$. We obtain analogous formulas and error bounds for approximations over contours in the complex plane, by use of a conformal map transformation taking $\mathscr{D}_d'$ onto some other domain $\mathscr{D}$. Some of the new results are rather surprising. For example, if f is analytic and bounded in the unit disc $f(\pm 1) \neq 0$, and if F is defined by $F(x) = (1 - x^2)^\alpha f(x)$, where $\alpha > 0$, then taking $h = \pi/(2\alpha N)^{1/2}$, yields

$$F(x) - \sum_{k=-N}^{N} F(\tanh(kh/2)) \frac{\sin\{(\pi/h)[\log((1+x)/(1-x)) - kh]\}}{(\pi/h)[\log((1+x)/(1-x)) - kh]} \tag{1.7}$$

$$= O(N^{1/2} e^{-\pi(N\alpha/2)^{1/2}}) \qquad \text{as} \qquad N \to \infty,$$

for all $x \in [-1, 1]$. This result should be compared with that of interpolating F over $[-1, 1]$ by a polynomial P_{2N} of degree $2N$ for which [Timan, A. F. "Theory of Approximation Functions of a Real Variable," Fitzmatgiz, Moscow, 1960]

$$\max_{(x \in [-1,1])} |F(x) - p_{2N}(x)| \geqslant c/N^{2\alpha}$$

for all $N > 0$, where c is a positive constant independent of N.

1. Introduction and Summary

Let f be a given function defined on the real line R. Whittaker's cardinal function, $C(f, h, x)$, is defined by

$$C(f, h, x) = \sum_{k=-\infty}^{\infty} f(kh) \operatorname{sinc}[(x - kh)/h] \tag{1.1}$$

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whenever this series converges. In (1.1), h is a positive constant, and

$$\operatorname{sinc} x = \sin \pi x / \pi x. \tag{1.2}$$

The function $C(f, h, \cdot)$ was discovered by E. T. Whittaker [17] and studied extensively by J. M. Whittaker (see, e.g., [18]).

Let us set

$$C_{M,N}(f, h, x) = \sum_{k=-M}^{N} f(kh) \operatorname{sinc} [(x - kh)/h] \tag{1.3}$$

$$T_{M,N}(f, h) = h \sum_{k=-M}^{N} f(kh). \tag{1.4}$$

The approximation $T_{M,N}$ is obtained by integrating $C_{M,N}$ over R. In 1949 Goodwin [4] discovered the incredible accuracy of the approximation $T_{M,N}$ of the integral of f over R, for those functions f which are analytic in the strip $\mathscr{D}_d' = \{x + iy: |y| \leqslant d\}$ and which converge rapidly to zero as $x \to \pm\infty$. As may be expected, $C_{M,N}(f, h, \cdot)$ is also a very accurate approximation of such a function f on R.

Many of the known properties of $C(f, h, \cdot)$ are described in [8]. In the present paper we briefly recall some of these, and we derive some others. We show, for example that the Hilbert transform of $C_{M,N}(f, h, \cdot)$ over R is a very accurate approximation of the Hilbert transform of an analytic function f of the type referred to above.

In addition, we derive some new analogous approximation formulas by use of conformal transformations of the region $\mathscr{D}_d'$ onto other regions $\mathscr{D}$. For example, by taking $\mathscr{D} = \{z = x + iy: |z| < 1\}$ we obtain formulas for interpolating, integrating, and evaluating the Hilbert transform of functions over $(-1, 1)$; by taking $\mathscr{D} = \{x + iy: x > 0\}$, we get analogous formulas over $(0, \infty)$.

We also obtain accurate error bounds, which enable us to identify classes of functions for which the formulas are very accurate. For example, if f is analytic in $\{x + iy: x > 0\}$, if

$$\lim_{c \to 0^+} \int_{-\infty}^{\infty} \left| \frac{F(c + iy)}{c + iy} \right| dy < \infty, \qquad \int_{-\pi/2}^{\pi/2} |f(Re^{i\theta})| \, d\theta \to 0 \qquad \text{as} \quad R \to \infty \tag{1.5}$$

and if $|f(x)| \leqslant Cx^{\alpha}/(1 + x^2)^{\alpha}$ on $[0, \infty]$, where C and α are positive constants, then by taking $h = \pi/(2\alpha N)^{1/2}$,

$$f(x) - \sum_{k=-N}^{N} f(e^{kh}) \operatorname{sinc} \left[\frac{\log x - kh}{h}\right] = O(N^{1/2} e^{-\pi(\alpha/2)^{1/2} N^{1/2}}) \tag{1.6}$$

as $N \to \infty$, uniformly for $x \in [0, \infty]$. Similarly, if f is analytic in $\{x + iy: x > 0\}$, if f satisfies

$$\lim_{c \to 0^+} \int_{-\infty}^{\infty} |f(c + iy)|\, dy < \infty, \qquad \int_{-\pi/2}^{\pi/2} |f(\mathrm{Re}^{i\theta})|\, d\theta \to 0 \quad \text{as} \quad R \to \infty, \tag{1.7}$$

and if $|f(x)| \leqslant Cx^{\alpha-1}/(1 + x^2)^{\alpha-1/2}$, for all $x \in (0, \infty)$, where C and α are positive constants, then by taking $h = \pi/(\alpha N)^{1/2}$,

$$\text{P.V.} \int_0^{\infty} \frac{F(t)}{t - x}\, dt - h \sum_{k=-N}^{N} \frac{F(xe^{kh+h/2})}{1 - e^{-kh-h/2}} = O(e^{-\pi\alpha^{1/2}N^{1/2}}) \tag{1.8}$$

as $N \to \infty$, for every fixed x in $(0, \infty)$.

Each of the above estimates is derived by first obtaining a Davis-type error bound [1, p. 345] and then minimizing this bound by expressing the step-size h as a function of N.

2. Properties of $C(f, h, x)$

Let us briefly recall some known properties of $C(f, h, x)$ (see [8]).

Definition 2.1. Let $B(h)$ denote the family of all functions $f \in L^2(R)$ such that f is an entire function of order $\leqslant 1$ and type $\leqslant \pi/h$, i.e.,

$$|f(z)| \leqslant Ce^{\pi|z|/h} \tag{2.1}$$

for all complex z where C is a constant, and

$$\|f\|_2^2 = \int_R |f(t)|^2\, dt < \infty. \tag{2.2}$$

Theorem 2.1 [8]. *If $f \in B(h)$, then*

$$f(z) = C(f, h, z) \tag{2.3}$$

for all complex z.

Theorem 2.2. *If $f \in B(h)$, then*

$$\lim_{n \to \infty} \int_{-nh}^{nh} f(t)\, dt - h \sum_{k=-n}^{n} f(kh) = 0. \tag{2.4}$$

Proof. This result follows directly from (2.3) upon integration of the cardinal series, and using the identity

$$\int_R \operatorname{sinc}(x/h)\, dx = h. \tag{2.5}$$

The relations

$$\begin{aligned} \int_R \operatorname{sinc}\left(\frac{x-mh}{h}\right) \operatorname{sinc}\left(\frac{x-nh}{h}\right) dx &= h \qquad \text{if} \quad m = n \\ &= 0 \qquad \text{if} \quad m \neq n \end{aligned} \tag{2.6}$$

where m and n are integers, lead to

THEOREM 2.3. *If $f \in B(h)$, then*

$$\int_R |f(t)|^2\, dt = h \sum_{k=-\infty}^{\infty} |f(kh)|^2. \tag{2.7}$$

The sequence

$$\{(1/h^{1/2}) \operatorname{sinc}((x-kh)/h)\}_{k=-\infty}^{\infty} \tag{2.8}$$

is therefore a complete orthonormal sequence in $B(h)$.

Let us next define the Hilbert transform Hf, by

$$(Hf)(x) = \frac{\text{P.V.}}{\pi i} \int_R \frac{f(t)}{t-x}\, dt \tag{2.9}$$

where $f \in L^p(R)$, $p \geqslant 1$, and where P.V. denotes the principal value. Let us assume that f is such that there exists a function F in $L^q(R)(p^{-1} + q^{-1} = 1)$, such that

$$f(x) = \int_R e^{ixt} F(t)\, dt, \tag{2.10}$$

and define Pf by

$$(Pf)(x) = \int_0^\infty e^{ixt} F(t)\, dt. \tag{2.11}$$

Then (see, e.g., [12])

$$Hf = 2Pf - f. \tag{2.12}$$

The identity

$$\operatorname{sinc}\left[\frac{x-kh}{h}\right] = \int_{-\pi/h}^{\pi/h} \left(\frac{h}{2\pi}\right) e^{-ikht} e^{ixt}\, dt \tag{2.13}$$

thus yields

$$P\left(\operatorname{sinc}\left[\frac{\cdot - kh}{h}\right]\right)(x) = \frac{(h/(2\pi))[e^{i\pi(x-kh)/h} - 1]}{i(x-kh)}, \tag{2.14}$$

so that

$$H\left(\operatorname{sinc}\left[\frac{\cdot - kh}{h}\right]\right)(x) = \frac{i\pi}{2h}(x-kh)\operatorname{sinc}^2\left[\frac{x-kh}{2h}\right]. \tag{2.15}$$

By collecting the above results, we get

THEOREM 2.4. *Let $f \in B(h)$. Then*

$$f(x) = C(f, h, x); \tag{2.16}$$

$$\begin{aligned} \int_R e^{ixt} f(t)\,dt &= h \sum_{k=-\infty}^{\infty} f(kh)\, e^{ikhx}, & |x| < \pi/h \\ &= 0, & |x| > \pi/h; \end{aligned} \tag{2.17}$$

$$(Pf)(x) = \frac{h}{2\pi i} \sum_{k=-\infty}^{\infty} f(kh)\left[\frac{e^{i\pi(x-kh)/h} - 1}{x-kh}\right]; \tag{2.18}$$

$$(Hf)(x) = \frac{i\pi}{2h} \sum_{k=-\infty}^{\infty} f(kh)(x-kh)\operatorname{sinc}^2\left[\frac{x-kh}{2h}\right], \tag{2.19}$$

where Pf and Hf are defined in (2.11) *and* (2.9), *respectively.*

The explicit form (2.19) was derived in [13].

3. INTERPOLATION OF ANALYTIC FUNCTIONS

DEFINITION 3.1. Let $B_d{}^p$, where $d > 0$, $p \geqslant 1$ denote the family of all functions f that are analytic in

$$\mathscr{D}_d{}' \equiv \{x + iy : |y| < d\} \tag{3.1}$$

such that

$$\int_{-d}^{d} |f(x+iy)|\,dy \to 0 \quad \text{as} \quad x \to \pm\infty, \tag{3.2}$$

and such that

$$N(f, p, \mathscr{D}_d{}') \equiv \lim_{y \to d^-} \left\{ \left(\int_R |f(x+iy)|^p\,dx \right)^{1/p} + \left(\int_R |f(x-iy)|^p\,dx \right)^{1/p} \right\} < \infty. \tag{3.3}$$

THEOREM 3.2. *Let $f \in B_d^p$, $p = 1$ or 2, and define $\epsilon(f)$ by*

$$\epsilon(f)(x) = f(x) - C(f, h, x) \tag{3.4}$$

where $-\infty \leqslant x \leqslant \infty$. Then[1]

$$\epsilon(f)(x) = \frac{\sin(\pi x/h)}{2\pi i} \int_R \left\{ \frac{f(t - id^-)}{(t - x - id)\sin(\pi(t - id)/h)} - \frac{f(t + id^-)}{(t - x + id)\sin(\pi(t + id)/h)} \right\} dt. \tag{3.5}$$

Moreover, if $f \in B_d^1$, then

$$\| \epsilon(f)\|_\infty \leqslant \frac{N(f, 1, \mathscr{D}_d')}{2\pi d \sinh(\pi d/h)}, \tag{3.6}$$

while if $f \in B_d^2$, then

$$\| \epsilon(f)\|_2 , 2(\pi d)^{1/2} \| \epsilon(f)\|_\infty \leqslant \frac{N(f, 2, \mathscr{D}_d')}{\sinh(\pi d/h)}. \tag{3.7}$$

Proof. We shall first get (3.5) by proceeding as in [5, 6, 8]. Let r and s be real, and let a contour L_n be defined by $L_n = \{r + is: | r | \leqslant (n + 1/2)h$ and $s = \pm d$, $r = \pm(n + 1/2)h$ and $| s | \leqslant d\}$, where n is a positive integer. Then L_n encloses the points $x = 0, \pm h, \pm 2h,..., \pm nh$. Hence we deduce from Cauchy's theorem, that if $-nh \leqslant x \leqslant nh$,

$$f(x) - \sum_{k=-n}^{n} f(kh) \operatorname{sinc}[(x - kh)/h)] = \lim_{c \to 1^-} \frac{\sin(\pi x/h)}{2\pi i} \int_{L_n} \frac{f(cz)dz}{(z - x)\sin(\pi z/h)}. \tag{3.8}$$

If u and v are real, the relations

$$| \sin(u + iv)| = [\sinh^2 v + \sin^2 u]^{1/2} \geqslant \sinh | v | \tag{3.9}$$

show that on the vertical segments of L_n, $| \sin[\pi((n + 1/2)h + is)/h]|^2 = \sinh^2(\pi s/h) + 1 \geqslant 1$; hence letting $n \to \infty$ in (3.8) and using (3.2) we get (3.5). Notice that by letting x depend upon n, $-n \leqslant x \leqslant n$, we may let $x \to \pm\infty$ along with $\pm n$.

In order to get (3.6), we proceed as in [5, 6, 8]. We simply use (3.9), as well as $| t - x \pm id | \geqslant d$ in (3.5).

We next proceed as in [12] to get (3.7). Consider the integral

$$I(x) = \int_R \frac{f(t - id^-)\, dt}{(t - x - id)\sin[\pi(t - id)/h]}. \tag{3.10}$$

[1] Here and elsewhere $\int_R F(t \pm id^-)\, dt \equiv \lim_{y \to d^-} \int_R F(t \pm iy)\, dt$.

Schwarz's inequality and (3.9) now yield

$$\begin{aligned}
| I(x)| &\leqslant \int_R \frac{| f(t - id^-)| \, dt}{[(t-x)^2 + d^2]^{1/2} \sinh(\pi d/h)} \\
&\leqslant \frac{1}{\sinh(\pi d/h)} \left(\int_R \frac{dt}{(t-x)^2 + d^2} \right)^{1/2} \left(\int_R | f(t - id^-)|^2 \, dt \right)^{1/2} \qquad (3.11) \\
&= \frac{(\pi/d)^{1/2}}{\sinh(\pi d/h)} \left(\int_R | f(t - id^-)|^2 \, dt \right)^{1/2}.
\end{aligned}$$

A similar treatment of the second integral in (3.5) leads to the $\| \epsilon(f)\|_\infty$ bound in (3.7).

If $\varphi \in L^2(R)$, the function Φ defined by

$$\Phi(x) = \int_R e^{ixt} \varphi(t) \, dt \tag{3.12}$$

is also in $L^2(R)$. Moreover, by Parseval's theorem,

$$\frac{1}{2\pi} \int_R | \Phi(x)|^2 \, dx = \int_R | \varphi(t)|^2 \, dt. \tag{3.13}$$

Furthermore, if $\Phi_+(x + iy)$ is defined for $y > 0$ by

$$\begin{aligned}
\Phi_+(x + iy) &= \int_0^\infty e^{ixt} e^{-yt} \varphi(t) \, dt \\
&= \frac{1}{2\pi i} \int_R \frac{\Phi(t) \, dt}{t - x - iy},
\end{aligned} \tag{3.14}$$

then by (3.13)

$$\begin{aligned}
\int_R | \Phi_+(x + iy)|^2 \, dx &= 2\pi \int_0^\infty e^{-2yt} | \varphi(t)|^2 \, dt \\
&\leqslant 2\pi \int_R | \varphi(t)|^2 \, dt \\
&= \int_R | \Phi(x)|^2 \, dx.
\end{aligned} \tag{3.15}$$

By applying these results to (3.10), we get

$$\begin{aligned}
\| I \|_2^2 &\leqslant (2\pi)^2 \int_R \left| \frac{f(t - id^-)}{\sin[\pi(t - id)/h]} \right|^2 dt \\
&\leqslant \frac{(2\pi)^2}{\sinh^2(\pi d/h)} \int_R | f(t - id^-)|^2 \, dt.
\end{aligned} \tag{3.16}$$

A similar treatment of the second integral in (3.5) leads to the $\| \epsilon(f)\|_2$ bound in (3.7).

This completes the proof of Theorem 3.2.

4. Approximate Integration of Analytic Functions

Let $f \in B_d{}^1$. If we integrate both sides of (3.4), we get

$$\eta(f) \equiv \int_R \epsilon(f)(x)\,dx = \int_R f(x)\,dx - h \sum_{k=-\infty}^{\infty} f(kh). \tag{4.1}$$

We substitute (3.5) into (4.1), interchange the order of integration, and use the identities

$$\frac{1}{2\pi i} \int_R \frac{\sin(\pi x/h)\,dx}{t - x \pm id} = \frac{1}{2i}\, e^{-\pi d/h} e^{\pm it\pi/h} \tag{4.2}$$

to get

$$\eta(f) = \frac{e^{-\pi d/h}}{2i} \int_R \left\{ \frac{f(t + id^-)\, e^{it\pi/h}}{\sin[\pi(t + id)/h]} - \frac{f(t - id^-)\, e^{-it\pi/h}}{\sin[\pi(t - id)/h]} \right\} dt. \tag{4.3}$$

Taking the absolute value, we thus arrive at

Theorem 4.1. *If $f \in B_d{}^1$, then*

$$|\eta(f)| \leqslant \frac{e^{-\pi d/h}}{2 \sinh(\pi d/h)} N(f, 1, \mathscr{D}_d{}'). \tag{4.4}$$

where $\eta(f)$ is defined as in (4.1), *and $N(\cdot)$ as in* (3.3).

The proof of (4.4) is similar to that in [5]. The bound (4.4) was obtained by a different procedure in [6, 8].

5. Approximation of Fourier and Hilbert Transforms

We make the approximation

$$\begin{aligned} \int_R e^{ixt} f(t)\,dt &\simeq h \sum_{k=-\infty}^{\infty} f(kh)\, e^{ikhx}, \qquad & |x| \leqslant \pi/h \\ &\simeq 0, & |x| > \pi/h \end{aligned} \tag{5.1}$$

for the Fourier transform of f, where $f \in B_d{}^1$. For each $|x| \leqslant \pi/h$, the function $g \in B_d{}^1$, where $g(t) = e^{ixt} f(t)$. By applying Theorem 4.1, we get

THEOREM 5.1. *If $f \in B_d^1$, then for all $x \in [-\pi/h, \pi/h]$,*

$$\left| \int_R e^{ixt} f(t)\, dt - h \sum_{k=-\infty}^{\infty} f(kh)\, e^{ikhx} \right| \leqslant \frac{N(f, 1, \mathscr{D}_d')}{2 \sinh(\pi d/h)} . \tag{5.2}$$

Let us next obtain two Hilbert transform approximations. We assume that $f \in B_d^p$, $p = 1$ or 2. If $u \in R$ then it is readily verified using (2.10)–(2.12), that

$$\frac{\text{P.V.}}{\pi i} \int_R \frac{\sin(\pi x/h)\, dx}{(t - x \pm id)(x - u)} = i \left\{ \frac{1}{t \pm id - u} [e^{\pm i(t \pm id)\pi/h} - \cos(\pi u/h)] \right\}. \tag{5.3}$$

Let Hf be defined as in (2.9). Taking the Hilbert transform of each side of (3.5) and using (2.15) and (5.3), we get

$$\begin{aligned} \frac{1}{2\pi} \int_R & \left\{ \frac{f(t - id^-)}{t - u - id} \left[\frac{e^{-i(t-id)\pi/h} - \cos(\pi u/h)}{\sin[(t - id)\pi/h]} \right] \right. \\ & \left. - \frac{f(t + id^-)}{t - u + id} \left[\frac{e^{i(t+id)\pi/h} - \cos(\pi u/h)}{\sin[(t - id)\pi/h]} \right] \right\} dt \\ & = (Hf)(x) - \frac{i\pi}{2h} \sum_{k=-\infty}^{\infty} f(kh)(x - kh) \operatorname{sinc}^2 \left[\frac{x - kh}{2h} \right]. \end{aligned} \tag{5.4}$$

By proceeding as in the proof of Theorem 3.2 and using the inequalities $| t - u \pm id | \geqslant d$, and

$$\left| \frac{e^{\pm i(t \pm id)\pi/h} - \cos(\pi u/h)}{\sin[(t \pm id)\pi/h]} \right| \leqslant \frac{e^{-\pi d/h} + 1}{\sinh(\pi d/h)} = \frac{e^{-\pi d/(2h)}}{\sinh(\pi d/(2h))}$$

we deduce the results of the following theorem, which we believe to be new.

THEOREM 5.2. *Let $\delta(f)$ be defined by*

$$\delta(f)(x) = (Hf)(x) - (HC(f, h, \cdot))(x), \tag{5.5}$$

where Hf is defined as in (2.9), *and $HC(f, h, \cdot)$ is explicitly expressed in* (2.19). *If $f \in B_d^2$, then*

$$2(\pi d)^{1/2} \| \delta f \|_\infty , \| \delta f \|_2 \leqslant \frac{e^{-\pi d/(2h)} N(f, 1, \mathscr{D}_d')}{\sinh(\pi d/(2h))} \tag{5.6}$$

while if $f \in B_d^1$, then

$$\| \delta f \|_\infty \leqslant \frac{e^{-\pi d/(2h)} N(f, 1, \mathscr{D}_d')}{2\pi d \sinh(\pi d/(2h))} \tag{5.7}$$

where $N(f, p, \mathscr{D}_d')$ is defined in (3.3).

We shall derive another approximation of Hf for the case when $f \in B_d{}^p$, $p = 1$ or 2, by proceeding as in [5]. An alternate expression for Hf is

$$(Hf)(x) = \frac{1}{2\pi i}\int_R \frac{f(x+t) - f(x-t)}{t}\,dt = \int_R g(t)\,dt \qquad (5.8)$$

where

$$g((t) = \frac{1}{2i\pi}\left\{\frac{f(x+t+h/2) - f(x-t-h/2)}{t+h/2}\right\}. \qquad (5.9)$$

and where $g \in B_d{}^1$. Setting

$$\eta(g) = \int_R g(t)\,dt - h\sum_{k=-\infty}^{\infty} g(kh) \qquad (5.10)$$

and using (4.3), we get

$$\eta(g) = \frac{e^{-\pi d/h}}{-2i}\int_R\left\{\frac{g(t-id^-)\,e^{-it\pi/h}}{\sin[\pi(t-id)/h]} - \frac{g(t+id^-)\,e^{it\pi/h}}{\sin[\pi(t+id)/h]}\right\}dt. \qquad (5.11)$$

By proceeding as in the proof Theorem 3.2, Eq. (3.6), we arrive at

THEOREM 5.3. *Let $f \in B_d{}^p$, $p = 1, 2$. The error $\eta(g)$ defined in* (5.10) *is also expressed as follows*:

$$\eta(g) = (Hf)(x) - \frac{1}{\pi i}\sum_{k=-\infty}^{\infty}\frac{f(x+kh+h/2)}{k+1/2}. \qquad (5.12)$$

Let $N(f, p, \mathscr{D}_d{}')$ be defined as in (3.3).

(a) *If $f \in B_d{}^1$, then*

$$\|\eta(g)\|_\infty \leqslant \frac{e^{-\pi d/h}}{2\pi d\,\sinh(\pi d/h)}\,N(f, 1, \mathscr{D}_d{}'). \qquad (5.13)$$

(b) *If $f \in B_d{}^2$, then*

$$\|\eta(g)\|_\infty \leqslant \frac{e^{-\pi d/h}}{2(\pi d)^{1/2}\sinh(\pi d/h)}\,N(f, 2, \mathscr{D}_d{}'). \qquad (5.14)$$

6. FORMULAS OBTAINED BY CONFORMAL MAPPING

Let φ be a conformal map of a simply connected domain $\mathscr{D}$ onto

$$\mathscr{D}' = \{x + iy : |y| < \pi/2\}. \qquad (6.1)$$

Let us denote the boundary of $\mathscr{D}$ by C, let a and $b \neq a$ be points of C, and let us assume that $\varphi(a) = -\infty$, $\varphi(b) = \infty$. We denote the inverse function, φ^{-1} by ψ, and set $\gamma = \psi((-\infty, \infty))$.

Let $B(\mathscr{D})$ denote the family of all functions F that are analytic in $\mathscr{D}$, such that

$$N(F, \mathscr{D}) = \lim_{C^1 \to C} \inf_{C^1 \subset \mathscr{D}} \int_{C^1} | F(z)\, dz | < \infty \tag{6.2}$$

and such that

$$\int_{\psi(u+L)} | F(z)\, dz | \to 0 \qquad \text{as} \quad u \to \pm\infty, \tag{6.3}$$

where

$$L = \{iy: | y | \leqslant \pi/2\}. \tag{6.4}$$

Let us define f by

$$f(w) = F(\psi(w))\, \psi'(w). \tag{6.5}$$

By recalling the definition of $N(f, 1, \mathscr{D}')$ in Eq. (3.3), it then follows that

$$N(F, \mathscr{D}) = N(f, 1, \mathscr{D}'). \tag{6.6}$$

THEOREM 6.1. *Let $F \in B(\mathscr{D})$. Let z_k be defined by $z_k = \psi(kh)$, $k = 0$, ± 1, ± 2,... .*

(a) *If $z \in \gamma = \psi((-\infty, \infty))$, then*

$$\delta(F)(z) \equiv \frac{F(z)}{\varphi'(z)} - \sum_{k=-\infty}^{\infty} \frac{F(z_k)}{\varphi'(z_k)} \operatorname{sinc} \left[\frac{\varphi(z) - kh}{h}\right] \tag{6.7}$$

is bounded as follows:

$$\| \delta(F) \|_\infty \leqslant \frac{N(F, \mathscr{D})}{\pi^2 \sinh(\pi^2/2h))}, \tag{6.8}$$

where $N(F, \mathscr{D})$ is defined in (6.2).

(b) *Let $w(F)$ be defined by*

$$w(F) \equiv \int_a^b F(z)\, dz - h \sum_{k=-\infty}^{\infty} \frac{F(z_k)}{\varphi'(z_k)}. \tag{6.9}$$

Then

$$| w(F) | \leqslant \frac{e^{-\pi^2/(2h)}}{2 \sinh(\pi^2/(2h))} N(F, \mathscr{D}). \tag{6.10}$$

Part (b) of Theorem 6.1 was proved by a different procedure in [14].

Part (a) of Theorem 6.1 is not in the most suitable form for purposse of interpolating F on γ. However, upon replacing F by $\varphi' F$, we immediately get the following result, which we believe to be new.

THEOREM 6.2. *Let G be defined by*

$$G(z) = F(z)\,\varphi'(z), \tag{6.11}$$

and let $G \in B(\mathscr{D})$. Then for all $z \in \gamma$,

$$\left| F(z) - \sum_{k=-\infty}^{\infty} F(z_k) \operatorname{sinc}\left[\frac{\varphi(z) - kh}{h}\right]\right| \leqslant \frac{N(G, \mathscr{D})}{\pi^2 \sinh(\pi^2/(2h))} \tag{6.12}$$

where $z_k = \psi(kh)$, and $N(G, \mathscr{D})$ is defined as $N(F, \mathscr{D})$ was, in (6.2).

Let us next derive a formula for approximating

$$(HF)(z) \equiv \frac{\text{P.V.}}{\pi i} \int_a^b \frac{F(t)}{t - z}\, dt \tag{6.13}$$

where the integration is along $\gamma = \psi((-\infty, \infty))$, and $z \in \gamma$. Setting

$$t = \psi(u), \qquad z = \psi(w) \tag{6.14}$$

we get

$$\begin{aligned}(HF)(z) &= \frac{\text{P.V.}}{\pi i} \int_R \frac{F(\psi(u))\,\psi'(u)\,du}{\psi(u) - \psi(w)} \\ &= \frac{1}{2\pi i} \int_R \left\{\frac{F(\psi(u+w))\,\psi'(u+w)}{\psi(u+w) - \psi(w)} - \frac{F(\psi(w-u))\,\psi'(w-u)}{\psi(w-u) - \psi(w)}\right\} du.\end{aligned} \tag{6.15}$$

Applying Theorem 4.1, we get

THEOREM 6.3. *Let $z \in \gamma$, and let $(HF)(z)$ be defined in* (6.13). *If $G \in B(\mathscr{D})$, where $G(t) = F(t)/(t - z)$,*

$$\left| (HF)(z) - \frac{h}{\pi i} \sum_{k=-\infty}^{\infty} \frac{F(\psi(\varphi(z) + kh + h/2))\,\psi'(\varphi(z) + kh + h/2)}{\psi(\varphi(z) + kh + h/2) - z} \right|$$

$$\leqslant \frac{e^{-\pi^2/(2h)}}{2\pi \sinh(\pi^2/(2h))} N(G, \mathscr{D}), \tag{6.16}$$

where $N(G, \mathscr{D})$ is defined as in (6.2).

This result is believed to be new.

7. Applications of Conformal Mappings

Let us begin with approximations over $(-1, 1)$. To this end, in the notation of Section 6, let F be analytic in

$$\mathscr{D} = \{z: |z| < 1\}, \tag{7.1}$$

and take $a = -1$, $b = 1$. Then φ defined by

$$w = \varphi(z) = \log((1+z)/(1-z)) \quad (\leftrightarrow z = \psi(w) = \tanh(w/2)) \tag{7.2}$$

maps $\mathscr{D}$ conformally onto $\mathscr{D}'$ (Eq. (6.1)) such that $\varphi(-1) = -\infty$, $\varphi(1) = \infty$. Paraphrasing Theorem 6.1, we get

Corollary 7.1. *Let F be analytic in $\mathscr{D}$ (Eq. (7.1)) and let*

$$N(F, \mathscr{D}) = \lim_{r\to 1^-} \int_0^{2\pi} |F(re^{i\theta})|\, d\theta < \infty \tag{7.3}$$

Then[2]

$$\left| \int_{-1}^{1} F(z)\, dz - h \sum_{k=-\infty}^{\infty} \frac{2e^{kh}}{(1+e^{kh})^2} F\left(\frac{e^{kh}-1}{e^{kh}+1}\right) \right| \leqslant \frac{e^{-\pi^2(2h)}}{2\sinh(\pi^2/(2h))} N(F, \mathscr{D}). \tag{7.4}$$

This result was obtained in [14]; related results were obtained independently in [11, 12, 15].

Paraphrasing Theorem 6.2, we get

Corollary 7.2. *Let F be analytic in $\mathscr{D}$ (Eq. (7.1), let φ be defined as in (7.2), G by*

$$G(z) = 2F(z)/(1 - z^2) \tag{7.5}$$

and let

$$N(G, \mathscr{D}) = \lim_{r\to 1^-} \int_0^{2\pi} |G(re^{i\theta})|\, d\theta < \infty. \tag{7.6}$$

Then for all $x \in [-1, 1]$, $x_k = \tanh(kh/2)$,

$$\left| F(x) - \sum_{k=-\infty}^{\infty} F(x_k) \operatorname{sinc}\left[\frac{\varphi(x) - kh}{h}\right] \right| \leqslant \frac{N(G, \mathscr{D})}{\pi^2 \sinh(\pi^2/(2h))}. \tag{7.7}$$

The approximation (7.7) is believed to be new.

[2] It is readily seen that if (7.3) is satisfied then the corresponding integrals analogous to (6.3) are also satisfied.

Similarly, paraphrasing Theorem 6.3, we get the following explicit result, which we believe to be new.

COROLLARY 7.3. *Let F be analytic in $\mathscr{D}$* (Eq. (7.1), *let G_x be defined by*

$$G_x(t) = \frac{F(t)}{t - x} \tag{7.8}$$

where $x \in (-1, 1)$, and let

$$N(G_x, \mathscr{D}) = \lim_{r \to 1^-} \int_0^{2\pi} | G_x(re^{i\theta})| \, d\theta < \infty. \tag{7.9}$$

then

$$\left| \frac{\text{P.V.}}{\pi i} \int_{-1}^{1} \frac{F(t)}{t - x} \, dt - \frac{h}{2\pi i} \sum_{k=-\infty}^{\infty} \frac{1 - x_k^2}{x_k(1 + xx_k)} F\left(\frac{x + x_k}{1 + xx_k}\right) \right|$$

$$\leqslant \frac{e^{-\pi^2/(2h)}}{2\pi \sinh(\pi^2/(2h))} N(G_x, \mathscr{D}), \tag{7.10}$$

where $x_k = \tanh(kh/2 + h/4)$.

Let us next derive analogous approximation formulas over the interval $(0, \infty)$. For this case, we assume that F is analytic in

$$\mathscr{D} = \{z: \operatorname{Re} z > 0\}, \tag{7.11}$$

and we take $a = 0$, $b = \infty$. Then φ and ψ are given by

$$w = \varphi(z) = \log z, \qquad z = \psi(w) = e^w. \tag{7.12}$$

Paraphrasing Theorem 6.1, we get

COROLLARY 7.4. *Let F be analytic in $\mathscr{D}$ (Eq.* (7.11)), *let*

$$N(F, \mathscr{D}) = \lim_{c \to 0^+} \int_{-\infty}^{\infty} | F(c + iy)| \, dy < \infty \tag{7.13}$$

and let

$$\int_{-\pi/2}^{\pi/2} | F(Re^{i\theta})| \, R \, d\theta \to 0 \qquad \text{as} \quad R \to \infty. \tag{7.14}$$

Then

$$\left| \int_0^{\infty} F(x) \, dx - h \sum_{k=-\infty}^{\infty} e^{kh} F(e^{kh}) \right| \leqslant \frac{e^{-\pi^2/(2h)}}{2 \sinh(\pi^2/(2h))} N(F, \mathscr{D}). \tag{7.15}$$

Results related to (7.10) were also obtained in [10, 11, 12, 14, 15].

Next, Theorem 6.2 yields an explicit interpolation formula over $[0, \infty]$, which we believe to be new.

COROLLARY 7.5. *Let F be analytic in $\mathscr{D}$ (Eq. (7.11)), let G be defined by*

$$G(z) = F(z)/z, \tag{7.16}$$

let

$$N(G, \mathscr{D}) = \lim_{c \to 0^+} \int_{-\infty}^{\infty} | G(c + iy)| \, dy < \infty, \tag{7.17}$$

and let

$$\int_{-\pi/2}^{\pi/2} | F(Re^{i\theta})| \, d\theta \to 0 \tag{7.18}$$

as $R \to \infty$. Then for all $x \in [0, \infty]$,

$$\left| F(x) - \sum_{k=-\infty}^{\infty} F(e^{kh}) \operatorname{sinc}\left[\frac{\log x - kh}{h}\right]\right| \leqslant \frac{N(G, \mathscr{D})}{\pi^2 \sinh(\pi^2/(2h))}. \tag{7.19}$$

We next apply Theorem 6.3, to get the following result, which we believe to be new.

COROLLARY 7.6. *Let F be analytic in $\mathscr{D}$ (Eq. (7.11)), let $x \in (0, \infty)$, let G_x be defined by*

$$G_x(t) = F(t)/(t - x) \tag{7.20}$$

and let

$$N(G_x, \mathscr{D}) = \lim_{c \to 0^+} \int_R | G_x(c + iy)| \, dy < \infty \tag{7.21}$$

and

$$\int_{-\pi/2}^{\pi/2} | F(Re^{i\theta})| \, d\theta \to 0 \qquad \text{as} \quad R \to \infty. \tag{7.22}$$

Then

$$\left| \frac{\text{P.V.}}{\pi i} \int_0^{\infty} \frac{F(t)}{t - x} \, dt - \frac{h}{\pi i} \sum_{k=-\infty}^{\infty} \frac{e^{kh+h/2} F(xe^{kh+h/2})}{e^{kh+h/2} - 1} \right|$$
$$\leqslant \frac{e^{-\pi^2/(2h)}}{2\pi \sinh(\pi^2/(2h))} N(G_x, \mathscr{D}). \tag{7.23}$$

8. RATE OF CONVERGENCE AS A FUNCTION OF THE NUMBER OF POINTS

In the application of the formulas derived in the provious sections, it is worthwhile to know the "price" one has to pay to achieve a certain accuracy and also the rate of convergence, in terms of the number of function evaluations that are required. We shall obtain such estimates, and thus complete the description of the class of functions for which the derived approximation formulas are expected to work well. The technique we use is that used in a particular case in [14]. Throughout this section, C_j, C^j, and α are positive constants, N is a positive integer, and $C_j(x)$ is a positive function of x, which is bounded on the open interval under consideration, but which may become unbounded at the end points of the interval.

Let us first consider the approximation over $(-\infty, \infty)$.

THEOREM 8.1. *Let $f \in B_d{}^p$, $p = 1$ or 2, $d > 0$ and let*

$$|f(x)| \leqslant C^1 e^{-\alpha|x|} \tag{8.1}$$

on $[-\infty, \infty]$.

(a) *Taking $h = (\pi d/(\alpha N))^{1/2}$, yields*

$$\left| f(x) - \sum_{k=-N}^{N} f(kh) \operatorname{sinc}\left[\frac{x-kh}{h}\right]\right| \leqslant C_1(N/\alpha)^{1/2}\, e^{-(\pi d\alpha N)^{1/2}} \tag{8.2}$$

and

$$\left|\frac{\text{P.V.}}{\pi i}\int_R \frac{f(t)}{t-x}\,dt - \frac{\pi i}{2h}\sum_{k=-N}^{N} f(kh)(x-kh)\operatorname{sinc}^2\left[\frac{x-kh}{h}\right]\right|$$
$$\leqslant C_2(N/\alpha)^{1/2}\, e^{-(\pi d\alpha N)^{1/2}} \tag{8.3}$$

for all $x \in [-\infty, \infty]$, and if $f \in B_d{}^1$, then

$$\left|\int_R e^{ixt} f(t)\,dt - h\sum_{k=-N}^{N} f(kh)\, e^{ikhx}\right| \leqslant \alpha^{-1} C_3 e^{-(\pi d\alpha N)^{1/2}} \tag{8.4}$$

for all $x \in [-\pi/h, \pi/h]$.

(b) *Taking $h = (2\pi d/(\alpha N))^{1/2}$ and $f \in B_d{}^1$, we get*

$$\left|\int_R f(x)\,dx - h\sum_{k=-N}^{N} f(kh)\right| \leqslant C_4 e^{-(2\pi d\alpha N)^{1/2}}, \tag{8.5}$$

while if $f \in B_d{}^p$, $p = 1$ or 2, then for all $x \in [-\infty, \infty]$,

$$\left|\frac{\text{P.V.}}{\pi i}\int_R \frac{f(t)}{t-x}\,dt - \frac{1}{\pi i}\sum_{k=-N-[x]}^{N-[x]} \frac{f(x+kh+h/2)}{k+1/2}\right| \leqslant \alpha^{-1} C_5 e^{-(2\pi d\alpha N)^{1/2}}, \tag{8.6}$$

where $[x]$ denotes the greatest integer $\leqslant x$.

Proof. The proofs of each of the above results are similar, and we therefore restrict ourselves to proving one case only, namely, the case of (8.2). The relation (8.5) was obtained previously in [14].

If f satisfies the conditions of Theorem 8.1(a), then by Theorem 3.2,

$$\begin{aligned}
&\left| f(x) - \sum_{k=-N}^{N}{}' f(kh) \operatorname{sinc}[(x-kh)/h] \right| \\
&\leqslant C_1' e^{-\pi d/h} + \sum_{k=N+1}^{\infty} \{|f(kh)| + |f(-kh)|\} \\
&\leqslant C_1' e^{-\pi d/h} + 2C^1 \sum_{k=N+1}^{\infty} e^{-\alpha kh} \\
&= C_1' e^{-\pi d/h} + 2C^1 (e^{-\alpha(N+1)h}/(1-e^{-\alpha h})) \\
&\leqslant C_1' e^{-\pi d/h} + (2C^1/\alpha h)\, e^{-\alpha N h},
\end{aligned} \tag{8.7}$$

where C_1' is a constant, and where we have used the inequality $\alpha h \leqslant e^{\alpha h} - 1$. Taking $h = (\pi d/(\alpha N))^{1/2}$ on the extreme right of (8.7) yields (8.2).

Let us next record estimates analogous to those of Theorem 8.1 for approximations over $(-1, 1)$ and over $(0, \infty)$.

THEOREM 8.2. (a) *Let F satisfy the conditions of Corollary* 7.2, *and on* $[-1, 1]$, *let* $|F(x)| \leqslant C^1(1-x^2)^\alpha$. *Then, taking* $h = \pi/(2\alpha N)^{1/2}$, $\varphi(x) = \log[(1+x)/(1-x)]$, *yields*

$$\left| F(x) - \sum_{k=-N}^{N} F(\tanh(kh/2)) \operatorname{sinc}\left(\frac{\varphi(x) - kh}{h}\right) \right| \leqslant C_1 (N/\alpha)^{1/2} e^{-\pi(\alpha N/2)^{1/2}} \tag{8.8}$$

for all $x \in [-1, 1]$.

(b) *Let F satisfy conditions of Corollary* 7.1, *and on* $(-1, 1)$, *let* $|F(x)| \leqslant C^2(1-x^2)^{\alpha-1}$. *Then taking* $h = \pi/(\alpha N)^{1/2}$,

$$\left| \int_{-1}^{1} F(x)\,dx - h \sum_{k=-N}^{N} \frac{2e^{kh}}{(1+e^{kh})^2} F\left(\frac{e^{kh}-1}{e^{kh}+1}\right) \right| \leqslant \alpha^{-1} C_2 e^{-\pi(\alpha N)^{1/2}}. \tag{8.9}$$

(c) *Let F satisfy the conditions of Corollary* 7.3, *and on* $(-1, 1)$, *let* $|F(x)| \leqslant C^3(1-x^2)^{\alpha-1}$. *Then by taking* $h = \pi/(\alpha N)^{1/2}$, $x_k = \tanh(kh/2 + h/4)$, *we get*

$$\begin{aligned}
&\left| \frac{\text{P.V.}}{\pi i} \int_{-1}^{1} \frac{F(t)}{t-x}\,dt - \frac{h}{2\pi i} \sum_{k=-N}^{N} \frac{1-x_k^2}{x_k(1+xx_k)} F\left(\frac{x+x_k}{1+xx_k}\right) \right| \\
&\leqslant \alpha^{-1} C_3(x)\, e^{-\pi(\alpha N)^{1/2}},
\end{aligned} \tag{8.10}$$

for all $x \in (-1, 1)$.

THEOREM 8.3. (a) *Let F satisfy the conditions of Corollary 7.5, and on $(0, \infty)$, let $|F(x)| \leqslant C^1 x^\alpha/(1 + x^2)^\alpha$. Then, by taking $h = \pi/(2\alpha N)^{1/2}$, we get*

$$\left| F(x) - \sum_{k=-N}^{N} F(e^{kh}) \operatorname{sinc}\left[\frac{\log x - kh}{h}\right]\right| \leqslant C_1(N/\alpha)^{1/2} e^{-\pi(\alpha N/2)^{1/2}} \quad (8.11)$$

for all $x \in [0, \infty]$.

(b) *Let F satisfy the conditions of Corollary (7.4), and on $(0, \infty)$, let $|F(x)| \leqslant C^2 x^{\alpha-1}/(1 + x^2)^\alpha$. Then, by taking $h = \pi/(\alpha N)^{1/2}$, we get*

$$\left| \int_0^\infty F(x)\, dx - h \sum_{k=-N}^{N} e^{kh} F(e^{kh}) \right| \leqslant \alpha^{-1} C_2 e^{-\pi(\alpha N)^{1/2}}. \quad (8.12)$$

(c) *Let F satisfy the conditions of Corollary 7.6, and on $(0, \infty)$, let $|F(x)| \leqslant C^3 x^{\alpha-1}/(1 + x^2)^{\alpha-1/2}$. Then, by taking $h = \pi/(\alpha N)^{1/2}$*

$$\left| \frac{\text{P.V.}}{\pi i} \int_0^\infty \frac{F(t)}{t - x}\, dt - \frac{h}{\pi i} \sum_{k=-N}^{N} \frac{e^{kh+h/2} F(x e^{kh+h/2})}{e^{kh+h/2} - 1} \right| \leqslant \alpha^{-1} C_3(x)\, e^{-\pi(\alpha N)^{1/2}} \quad (8.13)$$

for all $x \in (0, \infty)$.

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